Stein fillable Seifert fibered 3–manifolds

ANA G. LECUONA, PAOLO LISCA

Dipartimento di Matematica "L. Tonelli",
Università di Pisa
Largo Bruno Pontecorvo 5
I-56127 Pisa, Italy

Email: lecuona@mail.dm.unipi.it and lisca@dm.unipi.it

Abstract We characterize the closed, oriented, Seifert fibered 3–manifolds which are oriented boundaries of Stein manifolds. We also show that for this class of 3–manifolds the existence of Stein fillings is equivalent to the existence of symplectic fillings.

AMS Classification 57R17; 53D10

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1 Introduction and statement of results

The most important dichotomy in 3–dimensional contact topology is the one introduced by Eliashberg between tight and overtwisted contact structures (see e.g. [5, 6]). Nowadays there are several different ways to prove that a contact structure is tight, but for a long time the only systematic way to construct tight contact structures on a closed 3–manifold Y was to show Y to be orientation preserving diffeomorphic to the oriented boundary of a Stein manifold and then appeal to a theorem of Eliashberg and Gromov [4, 10]. This naturally led to the question of which 3–manifolds carry tight contact structures, as well as to the related question of which 3–manifolds admit *Stein fillings*, i.e. are orientation preserving diffeomorphic to the boundary of a Stein manifold. The first example of an oriented 3–manifold admitting no Stein fillings was provided in [11], and infinitely many examples were found in [14, Theorem 4.2] and [16, Proposition 4.1]. While the classification of the closed, Seifert fibered 3–manifolds carrying tight contact structures was recently achieved [17], the classification of the Stein fillable ones was still missing. The purpose of the present paper is to fill this gap. Our main result, Theorem 1.5 below, identifies explicitely the family of closed, oriented, Seifert fibered 3–manifolds which are orientation preserving diffeomorphic to the boundary of a Stein manifold.

We need some preliminaries in order to state our results. Eliashberg [3] proved that smooth, even–dimensional manifolds carrying Stein structures can be characterized as having suitable handle decompositions. Gompf [9, Theorem 5.4] elaborated on Eliashberg's result to show that a closed, oriented, Seifert fibered 3–manifold Y admits a Stein filling unless Y is orientation preserving diffeomorphic to the oriented 3–manifold $Y(e_0; r_1, \ldots, r_k)$ given by the surgery description of Figure 1, where

$$e_0 = -1, \quad k \geq 3 \quad \text{and} \quad 1 > r_1 \geq \cdots \geq r_k > 0.$$

Gompf also discovered a sufficient condition for the existence of Stein fillings of

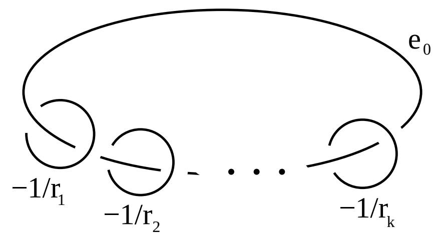

Figure 1: The Seifert 3–manifold $Y(e_0; r_1, \ldots, r_k)$

a 3–manifold of the form $Y(-1; r_1, \ldots, r_k)$. We describe and use this condition in Section 2.

Definition 1.1 A k–tuple $(r_1, \ldots, r_k) \in (\mathbb{Q} \cap (0,1))^k$ with $k \geq 3$ and $r_1 \geq r_2 \geq \cdots \geq r_k$ is *realizable* if there exist coprime integers $n > h > 0$ such that

$$\frac{h}{n} > r_1, \quad \frac{n-h}{n} > r_2, \quad \text{and} \quad \frac{1}{n} > r_3, \ldots, r_k.$$

Definition 1.2 A closed, oriented, Seifert fibered 3–manifold is *of special type* if it is orientation preserving diffeomorphic to $Y(-1; r_1, \ldots, r_k)$, where $k \geq 3$, $1 > r_1 \geq r_2 \geq \cdots \geq r_k > 0$ and the following conditions both hold:

(1) $(r_1, \ldots, r_k)$ is not realizable;

(2) $r_1 + \cdots + r_k > 1 > r_1 + r_2$.

In Section 2 we use Gompf's sufficient condition for the existence of Stein fillings of $Y(-1; r_1, \ldots, r_k)$ to establish the following:

Theorem 1.3 *Let Y be a closed, oriented, Seifert fibered 3–manifold which is not of special type. Then, Y is orientation preserving diffeomorphic to the boundary of a Stein surface.*

Recall (see e.g. [6]) that a *symplectic filling* of a contact 3–manifold (Y, ξ) is a pair (X, ω), where X is a smooth 4–manifold with boundary oriented by a symplectic form $\omega \in \Omega^2(X)$, and such that there is an orientation preserving diffeomorphism $\varphi\colon Y \to \partial X$ with $\omega|\varphi_*(\xi) \neq 0$ at each point of ∂X. A Stein filling is a symplectic filling but the converse is not true, because there are examples of symplectically fillable contact 3–manifolds which are not Stein fillable [7]. Similarly, there exist several examples of tight, contact Seifert fibered 3–manifolds which are not symplectically fillable [13, 15, 14, 8]. In Section 3 we apply Donaldson's theorem on the intersection forms of definite 4–manifolds to prove the following:

Theorem 1.4 *A closed, oriented, Seifert fibered 3–manifold of special type admits no symplectic fillings.*

Combining Theorems 1.3 and 1.4 immediately gives our main result:

Theorem 1.5 *Let Y be a closed, oriented, Seifert fibered 3–manifold. Then, the following conditions are equivalent:*

(1) *Y admits Stein fillings;*

(2) *Y admits symplectic fillings;*

(3) *Y is not of special type.*

Proof A Stein filling is a symplectic filling, therefore (1) implies (2). By Theorem 1.4 (2) implies (3) and by Theorem 1.3 (3) implies (1). □

Theorem 1.5 implies Corollary 1.6 below, which shows that the condition that a 3–manifold is Seifert fibered of special type can be reformulated in terms of open book decompositions. Recall that an open book decomposition of a closed 3–manifold is called *positive* if its monodromy can be written as a product of right–handed Dehn twists. Loi and Piergallini [19, Theorem 4] proved that a smooth, closed, oriented 3–manifold Y is the boundary of a Stein surface if and only if Y admits a positive open book decomposition. Combining this result with Theorem 1.5 immediately yields the following:

Corollary 1.6 *A closed, oriented, Seifert fibered 3–manifold Y admits a positive open book decomposition if and only if Y is not of special type.* □

We would like to end this introduction with a few remarks about "how special" Seifert 3–manifolds of special type are. First, it should be clear from the definition that there exist infinitely many oriented Seifert 3–manifolds of special type. Indeed, infinitely many examples of closed, oriented Seifert fibered 3–manifolds without symplectic fillings are known ([14, Theorem 4.2] and [16, Proposition 4.1]). According to Theorem 1.5 such examples must be all of special type, and in fact it can be easily verified that they are of special type. The infinitely many oriented Seifert 3–manifolds which do not carry tight contact structures [16, 17] are also of special type. Second, if $Y = Y(-1; r_1, \ldots, r_k)$ is of special type then $-Y$ is not, because it is of the form $Y(-1+k; 1-r_k, \ldots, 1-r_1)$. This is consistent with the general fact that each oriented Seifert fibered 3–manifold admits a Stein filling after possibly reversing its orientation [9, Corollary 5.5(a)]. Also, Condition (1) from Definition 1.2 together with the results of [18] imply that an oriented Seifert fibered 3–manifold of special type is an L–space in the sense of [23, Definition 1.1]. (See [18] for several different characterizations of Seifert fibered L–spaces). In fact, it follows from [18] that if Y is an oriented Seifert fibered 3–manifold which is an L–space, then after possibly reversing orientation Y is of the form $Y(e_0; r_1, \ldots, r_k)$, with $k \geq 3$, $1 > r_1 \geq \cdots \geq r_k > 0$ and either (i) $e_0 \geq 0$ or (ii) $e_0 = -1$ and $(r_1, \ldots, r_k)$ is not realizable. Therefore, the oriented Seifert fibered 3–manifolds of special type are precisely the oriented, Seifert fibered L–spaces of the form $Y(-1; r_1, \ldots, r_k)$ with $k \geq 3$, $1 > r_1 \geq \cdots \geq r_k > 0$ and $(r_1, \ldots, r_k)$ satisfying Condition (2) from Definition 1.2.

The organization of the paper is straightforward: in Section 2 we prove Theorem 1.3 and in Section 3 we prove Theorem 1.4.

2 Existence of Stein fillings

The purpose of this section is to prove Theorem 1.3. We start with recalling Gompf's sufficient condition from [9] for the existence of a Stein filling of $Y(-1; r_1, \ldots, r_k)$.

Given a rational number $r \in \mathbb{Q}$ we define an integer $[\![r]\!] \in \mathbb{Z}$ by setting $r = [\![r]\!] + \mathrm{frac}(r)$, where $\mathrm{frac}(r) \in [0, 1)$. Define

$$r_i' := -\frac{1}{r_i}, \quad i = 1, \ldots, k.$$

Let $s \in (-\infty, -1)$ be such that $\frac{1}{s} := -1 - \frac{1}{r_1'}$. If $s \neq r_2'$ then it is easy to check that there is a map

$$A\colon \mathbb{Q} \cup \{\infty\} \to \mathbb{Q} \cup \{\infty\}$$

of the form $A(r) = \frac{c+dr}{a+br}$, such that:

$$ad - bc = \pm 1, \tag{2.1}$$

$$A(s) \in (-1, 0], \tag{2.2}$$

$$A(r_2') \in [-\infty, -1). \tag{2.3}$$

Let

$$t := \begin{cases} 0 & \text{if } A(0) \in [0, +\infty] \\ \frac{1}{A(s)} & \text{if } A(0) \in [-1, 0) \\ A(r_2') & \text{if } A(0) \in (-\infty, -1). \end{cases}$$

Set $M := \max(|a|, |c|)$, $m := \min(|a|, |c|)$ and

$$n_A(r_1', r_2') := -m([\![t]\!] + 1) - M.$$

Finally, let

$$n(r_1', r_2') := \begin{cases} 0 & \text{if } s = r_2', \\ \sup_A n_A(r_1', r_2') & \text{if } s \neq r_2'. \end{cases}$$

Gompf [9] shows that $Y(-1; r_1, \ldots, r_k)$ is the boundary of a Stein surface if:

$$n(r_1', r_2') > r_3', \ldots, r_k'. \tag{2.4}$$

Observe that when $s = r_2'$ Condition (2.4) is automatically satisfied.

In order to prove Theorem 1.3 we need two results. The first result is Theorem 2.1, which establishes the existence of a Stein filling for $Y(-1; r_1, \ldots, r_k)$ under the assumption that the k–tuple $(r_1, \ldots, r_k)$ is *realizable*, that is to say that there exist coprime integers $n > h > 0$ such that $r_1 < \frac{h}{n}$, $r_2 < \frac{n-h}{n}$ and $r_3, \ldots, r_k < \frac{1}{n}$.

Theorem 2.1 *Suppose that $k \geq 3$, $1 > r_1 \geq r_2 \geq \cdots \geq r_k > 0$ and $(r_1, \ldots, r_k)$ is realizable. Then, $Y(-1; r_1, \ldots, r_k)$ is orientation preserving diffeomorphic to the boundary of a Stein surface.*

Proof Recall that we defined $r_i' := -\frac{1}{r_i}$, $i = 1, \ldots, k$. We will prove that there is a map $A\colon \mathbb{Q} \cup \{\infty\} \to \mathbb{Q} \cup \{\infty\}$ satisfying Properties (2.1), (2.2), (2.3) above, and such that

$$n_A(r_1', r_2') > r_3', \ldots r_k'.$$

In view of Gompf's condition (2.4), this clearly suffices to prove the statement.

By the realizability assumption, there is a positive integer n_0 such that, for some integer h_0 coprime with n_0 and satisfying $n_0 > h_0 > 0$ we have

$$r'_2 < -\frac{n_0}{h_0} < s \quad \text{and} \quad -n_0 > r'_3, \dots r'_k.$$

Denote by n the smallest positive integer such that, for some integer h coprime with n with $n > h > 0$, we have

$$n \le n_0 \quad \text{and} \quad r'_2 < -\frac{n}{h} < s.$$

Notice that, since $n \le n_0$, $-n \ge -n_0 > r'_3, \dots, r'_k$. Moreover, n and h being coprime, there exist $a, b \in \mathbb{Z}$ such that

$$ah - bn = 1. \tag{2.5}$$

If the pair (a, b) solves Equation (2.5), so does the pair $(a+zn, b+zh)$ for each $z \in \mathbb{Z}$. Therefore, we can choose a solution (a, b) such that $0 \le a < n$. Indeed, since $a = 0$ would imply $n = 1$, which is not the case because $n > h > 0$, we can assume $0 < a < n$. From Equation (2.5) we get

$$b = \frac{ah}{n} - \frac{1}{n},$$

hence $-1/n < b < h - 1/n$, which is equivalent to

$$0 \le b < h.$$

Soon it will be convenient to have $b > 0$, therefore we deal now with the special case $b = 0$. By Equation (2.5), $b = 0$ implies $a = h = 1$, therefore $r'_2 < -n < s$. Moreover, by the minimality of n we must have $-(n-1) \ge s$. Define the map A by

$$A(r) := r + n - 1.$$

This map is of the form $\frac{c+dr}{a+br}$ with $a = 1$, $b = 0$, $c = n-1$ and $d = 1$, therefore $ad - bc = 1$. Clearly A is monotone increasing, $A(-n) = -1$ and $A(-n+1) = 0$. Therefore $A(r'_2) \in (-\infty, -1)$ and $A(s) \in (-1, 0]$. Therefore A satisfies the required Properties (2.1), (2.2) and (2.3). Since $A(0) = n - 1 \in [0, +\infty]$ we have $t = 0$, $m = 1$ and $M = n - 1$, therefore

$$n_A(r'_1, r'_2) = -m - M = -1 - n + 1 = -n > r'_3, \dots, r'_k.$$

From now on we assume $0 < b < h$. Observe that Equation (2.5) is equivalent to

$$\frac{a}{b} = \frac{n}{h} + \frac{1}{hb},$$

which implies

$$-\frac{a}{b} < -\frac{n}{h}.$$

In fact, by our choice of n we must have

$$-\frac{a}{b} \leq r_2'.$$

Now we define A by

$$A(r) := \frac{(n-a)+(h-b)r}{a+br} = -1 + \frac{n+hr}{a+br}.$$

For this map we have $c = n - a$ and $d = h - b$, therefore

$$ad - bc = a(h-b) - b(n-a) = ah - bn = 1.$$

The map A is monotone increasing for every $r \neq -\frac{a}{b}$, because

$$\frac{dA}{dr}(r) = \frac{1}{(a+br)^2}.$$

Equation (2.5) implies $-\frac{n-a}{h-b} > -\frac{n}{h}$, thus by the choice of n we have $s \leq -\frac{n-a}{h-b}$. We conclude

$$A(-\frac{a}{b}) = -\infty \leq A(r_2') < A(-\frac{n}{h}) = -1 < A(s) \leq A(-\frac{n-a}{h-b}) = 0.$$

Therefore A satisfies Properties (2.1), (2.2) and (2.3). Since

$$A(0) = -1 + \frac{n}{a} = \frac{n-a}{a} \in (0, +\infty)$$

we have $t = 0$, thus

$$n_A(r_1', r_2') = -m - M = -|a| - |c| = -a - (n-a) = -n > r_3', \ldots, r_k'.$$

□

We can now move on to the second result needed for the proof of Theorem 1.3, that is Theorem 2.5 below. This result will establish the existence of a Stein filling for $Y(-1; r_1, \ldots, r_k)$ under the assumption $r_1 + r_2 \geq 1$. The proof of Theorem 1.3 will then follow combining Theorems 2.1 and 2.5.

Consider the standard Farey tessellation of the hyperbolic plane. Figure 2 illustrates some of the arcs of the tessellation with both endpoints in the interval $[-\infty, -1]$. We shall refer to any such arc with endpoints $\alpha < \beta$ as to the *Farey arc* $\widehat{\alpha\beta}$.

Observe that, given a Farey arc $\widehat{\alpha\gamma}$, there is a unique point β such that $\alpha < \beta < \gamma$ and there exist Farey arcs $\widehat{\alpha\beta}$ and $\widehat{\beta\gamma}$. In what follows, we shall refer to the unique point β as to the *middle point* of $\widehat{\alpha\gamma}$, and denote it by $m(\alpha, \gamma)$.

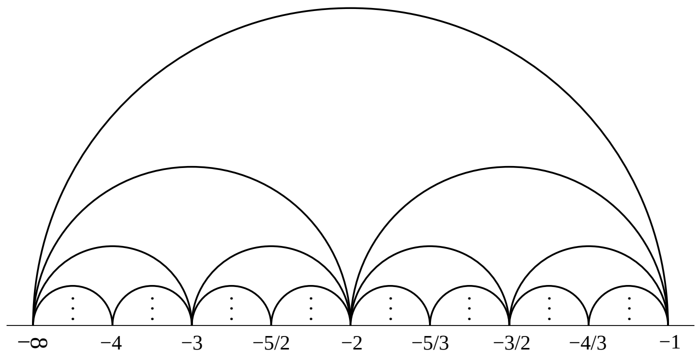

Figure 2: Arcs of the Farey tessellation

Lemma 2.2 *Let $s, r'_2 \in (-\infty, -1)$, with $s < r'_2$. Then, there exist a Farey arc $\widehat{\alpha\gamma}$ with middle point $\beta = m(\alpha, \gamma)$ satisfying*

$$-\infty \leq \alpha < s \leq \beta \leq r'_2 < \gamma \leq -1.$$

In other words, there is a configuration of Farey arcs as in Figure 3(A), 3(B) or 3(C).

Proof We provisionally define $\alpha = -\infty$ and $\gamma = -1$. If $s \leq \beta = m(\alpha, \gamma) \leq r'_2$ then α, β and γ already satisfy the statement and the lemma is proved. Otherwise we have either $s > \beta$ or $r'_2 < \beta$ (but not both, because $s < r'_2$). If $s > \beta$ we redefine $\alpha = \beta$, while if $r'_2 < \beta$ we redefine $\gamma = \beta$, and in both cases we set β equal to the new middle point $m(\alpha, \gamma)$. As before, if $s \leq \beta = m(\alpha, \gamma) \leq r'_2$ we are done, otherwise either $s > \beta$ or $r'_2 < \beta$ (but not both). Continuing in this fashion, after a finite number of steps we necessarily arrive at a configuration satisfying the statement of the lemma. $\square$

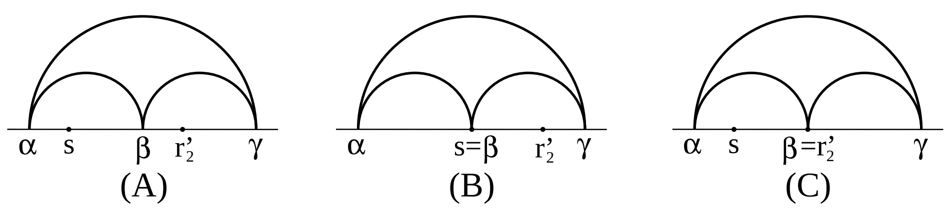

Figure 3: The configurations of Farey arcs of Lemma 2.2.

Lemma 2.3 *Let $\widehat{\alpha\beta}$ be a Farey arc with $\alpha, \beta \in [-\infty, -1]$, and let $s \in (\alpha, \beta)$. Then, there exist Farey arcs $\widehat{\alpha\beta'}$ and $\widehat{\alpha'\beta}$ such that*

$$m(\alpha, \beta') \leq s < \beta' \leq \beta \quad \text{and} \quad \alpha \leq \alpha' < s \leq m(\alpha', \beta).$$

In other words, there are configurations of Farey arcs as in Figure 4(A) and 4(B).

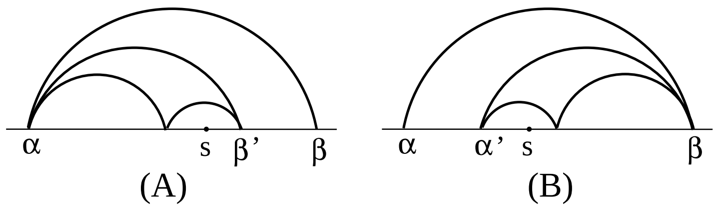

Figure 4: The configurations of Farey arcs of Lemma 2.3.

Proof We only prove the existence of the arc $\widehat{\alpha\beta'}$. The existence of the arc $\widehat{\alpha'\beta}$ can be established in the same way. If $m(\alpha, \beta) \leq s$ we define $\beta' = \beta$, and the statement is proved. If $s < m(\alpha, \beta)$, we define (temporarily) $\beta' = m(\alpha, \beta)$. If $m(\alpha, \beta') \leq s$ the arc $\widehat{\alpha\beta'}$ satisfies the statement. Otherwise $s < m(\alpha, \beta')$, we redefine $\beta' = m(\alpha, \beta')$ and we keep going in the same way. In a finite number of steps we are bound to find the arc $\widehat{\alpha\beta'}$ with the stated property. $\square$

Lemma 2.4 *Let $s, r_2' \in (-\infty, -1)$, with $s < r_2'$. Then, there exist Farey arcs $\widehat{\alpha\beta}$, $\widehat{\beta\gamma}$, $\widehat{\gamma\delta}$, $\widehat{xy}$, with $\widehat{xy} \in \{\widehat{\alpha\gamma}, \widehat{\beta\delta}\}$, satisfying*

$$-\infty \leq \alpha < s \leq \beta < \gamma \leq r_2' < \delta \leq -1.$$

In other words, there is a configuration of Farey arcs as in Figure 5(A) or 5(B).

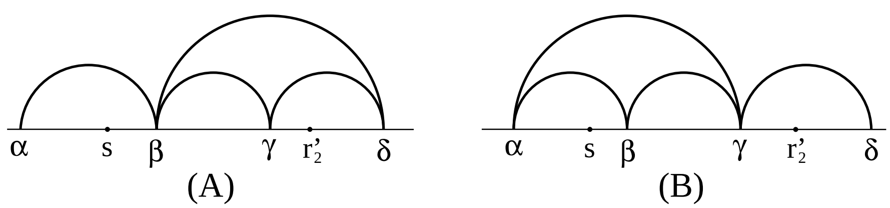

Figure 5: The configurations of Farey arcs of Lemma 2.4.

Proof By Lemma 2.2 there is a configuration of Farey arcs as in Figure 3(A), 3(B) or 3(C). If 3(A) holds we can apply Lemma 2.3 to the point s and the arc $\widehat{\alpha\beta}$ of Figure 3(A) to find a configuration as in Figure 4(B). If we set $m = m(\alpha', \beta)$, the Farey arcs $\widehat{\alpha' m}$, $\widehat{m\beta}$, $\widehat{\alpha'\beta}$ and $\widehat{\beta\gamma}$ provide a configuration as in Figure 5(B). If 3(B) holds we can apply Lemma 2.3 to the point r_2' and the arc $\widehat{\beta\gamma}$ of Figure 3(B) to find a configuration as in Figure 4(A). In other

words, there exists a Farey arc $\widehat{\beta\beta'}$ such that $m(\beta,\beta') \le r_2' < \beta'$. Setting $m = m(\beta,\beta')$, the Farey arcs $\widehat{\alpha\beta}$, $\widehat{\beta m}$, $\widehat{m\beta'}$ and $\widehat{\beta\beta'}$ provide a configuration as in Figure 5(A). Finally, if 3(C) holds we can apply Lemma 2.3 to the point s and the arc $\widehat{\alpha\beta}$ of Figure 3(C) to find a configuration as in Figure 4(B). If we set $m = m(\alpha',\beta)$, the Farey arcs $\widehat{\alpha' m}$, $\widehat{m\beta}$, $\widehat{\alpha'\beta}$ and $\widehat{\beta\gamma}$ provide a configuration as in Figure 5(B). □

Theorem 2.5 *Suppose that $k \ge 3$, $1 > r_1 \ge r_2 \ge \cdots \ge r_k > 0$ and $r_1 + r_2 \ge 1$. Then, $Y(-1; r_1, \ldots, r_k)$ is orientation preserving diffeomorphic to the boundary of a Stein surface.*

Proof It is easy to check that the condition $r_1 + r_2 \ge 1$ is equivalent to $s \le r_2'$. If $s = r_2'$ Condition (2.4) is automatically satisfied, therefore we may assume $s < r_2'$. By Lemma 2.4 there is a configuration of Farey arcs as in Figure 5(A) or 5(B). Let us suppose that the first case occurs. In view of the action of $PSL(2,\mathbb{Z})$ on the Farey tessellation, there is a unique map $A\colon \mathbb{Q}\cup\{\infty\} \to \mathbb{Q}\cup\{\infty\}$ of the form $A(r) = \frac{c+dr}{a+br}$ satisfying Condition (2.1) and such that $A(\beta) = 0$, $A(\gamma) = \infty$ and $A(\delta) = -1$. By construction, A satisfies Conditions (2.2) and (2.3) as well, $c/a = A(0) \in (-1,0)$ and $A(s) \in (-1,0)$. According to Gompf's condition (2.4), in order to prove that $Y(-1; r_1, \ldots, r_k)$ carries Stein fillable contact structures it suffices to show that

$$n_A(r_1', r_2') = -m([\![t]\!] + 1) - M \ge -1,$$

where $t = 1/A(s) \in (-\infty, -1)$, $M = \max(|a|,|c|)$ and $m = \min(|a|,|c|)$. Observe that, since $A(0) = c/a \in (-1,0)$, $M = |a|$ and $m = |c|$. Condition (2.1) implies that if $m = 0$ then $M = 1$, therefore we may assume without loss of generality that $m > 0$. An easy calculation shows that the condition $-m([\![t]\!] + 1) - M \ge -1$ is equivalent to

$$[\![t]\!] + 1 - \frac{1}{m} \le -\frac{M}{m} = \frac{1}{A(0)}.$$

In order to prove this inequality it suffices to show that there is an integer N strictly greater than $1/A(s)$ and less than or equal to $1/A(0)$, i.e. such that

$$[\![t]\!] + 1 \le N \le \frac{1}{A(0)}.$$

This condition is satisfied if and only if there exists a Farey arc $\widehat{\infty x}$ with $1/A(s) < x \le 1/A(0)$ or, equivalently, if and only if there exists a Farey arc $\widehat{y0}$ with $A(0) \le y < A(s)$. Setting $y := A(\alpha)$, such an arc is provided by the image under A of the Farey arc $\widehat{\alpha\beta}$ from Lemma 2.4, because by construction

$A(\beta) = 0$ and $0 < \infty = -\infty \leq \alpha < s$. This concludes the proof under the assumption that when at the beginning of the argument we apply Lemma 2.4 we end up with a configuration of Farey arcs as in Figure 5(A). In case the configuration is the one given by Figure 5(B) we can argue in a similar way, so we just describe the steps where there is a difference. We choose the unique map A such that $A(\alpha) = -1$, $A(\beta) = 0$ and $A(\gamma) = +\infty = -\infty$. Then, $c/a = A(0) \in (-\infty, -1)$, $t = A(r_2') \in (-\infty, -1)$, $M = |c|$, $m = |a|$ and as before we may assume without loss that $m > 0$. The same calculation as in the previous case shows that Gompf's condition is equivalent to

$$[\![A(r_2')]\!] + 1 - \frac{1}{m} \leq -\frac{M}{m} = A(0).$$

As before, this condition is satisfied if there exists a Farey arc $\widehat{-\infty z}$ with $A(r_2') < z \leq A(0)$. Setting $z := A(\delta)$, such an arc is provided by the image under A of the Farey arc $\widehat{\gamma\delta}$ from Lemma 2.4, because by construction $A(\gamma) = -\infty$ and $r_2' < \delta \leq -1 < 0$. □

We are now ready to prove Theorem 1.3. We restate the result for the reader's convenience:

Theorem 1.3 Let Y be a closed, oriented, Seifert fibered 3–manifold which is not of special type. Then, Y is orientation preserving diffeomorphic to the boundary of a Stein surface.

Proof Gompf showed [9, Theorem 5.4] that a closed, oriented, Seifert fibered 3–manifold Y admits a Stein filling unless it is of the form $Y(-1; r_1, \dots, r_k)$ with $k \geq 3$ and $1 > r_1 \geq r_2 \geq \cdots \geq r_k > 0$, and by [1, 20] (see [20, pp. 157–158]), if $r_1 + \cdots + r_k \leq 1$ then $(r_1, \dots, r_k)$ is realizable. Thus, applying Theorem 2.1 we conclude that if Y is not of special type then Y is diffeomorphic to the boundary of a Stein surface unless Y is of the form $Y(-1; r_1, \dots, r_k)$ with $k \geq 3$, $1 > r_1 \geq r_2 \geq \cdots \geq r_k > 0$ and $r_1 + r_2 \geq 1$. But the latter conditions are precisely the assumptions of Theorem 2.5, therefore if Y is not of special type then Y is necessarily the boundary of a Stein surface. □

3 Nonexistence of symplectic fillings

The purpose of this section is to establish Theorem 1.4. We start with some preliminaries, then we prove three auxiliary lemmas. After that we prove Theorem 1.4.

Let $Y = Y(e_0; r_1, ..., r_k)$ denote the oriented, Seifert fibered 3–manifold given by the surgery description of Figure 1, where $e_0 \in \mathbb{Z}$, $r_i \in (0,1) \cap \mathbb{Q}$ and $r_1 \geq r_2 \geq \cdots \geq r_k$. The oriented 3–manifold Y is the oriented boundary of the 4–dimensional plumbing P_Γ of D^2–bundles over 2–spheres described by the star–shaped weighted graph Γ with k legs illustrated in Figure 6.

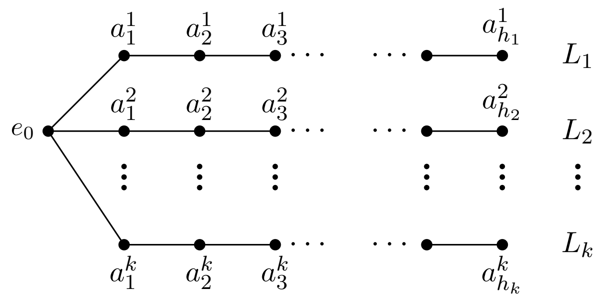

Figure 6: The weighted star–shaped graph Γ

The weights of the vertices in the leg L_i, which form the string $(a_1^i, ..., a_{h_i}^i)$, are given by the unique continued fraction expansion

$$-\frac{1}{r_i} = [a_1^i, ..., a_{h_i}^i] := a_1^i - \cfrac{1}{a_2^i - \cfrac{1}{\ddots \; a_{h_i-1}^i - \cfrac{1}{a_{h_i}^i}}}, \quad i = 1, \ldots, k$$

such that $a_j^i \leq -2$ for every j.

We can associate to Γ the intersection lattice $(\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$ of the plumbing P_Γ. In the proof of Theorem 1.4 we will show that if Y admits a symplectic filling then the intersection lattice of the plumbing associated to $-Y$ admits an isometric embedding into a standard diagonal lattice. Our present aim will be to prepare the ground for Lemma 3.3, which shows that under a certain assumption such an embedding does not exist.

We shall need Riemenschneider's point rule [24], which we now recall. Let $p > q > 0$ be coprime integers, and suppose

$$-\frac{p}{q} = [a_1, \ldots, a_l], \; a_i \leq -2, \quad -\frac{p}{p-q} = [b_1, \ldots, b_m], \; b_j \leq -2.$$

Then, the coefficients $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$ are related by a diagram of the

form

$$\begin{array}{ccccc} \bullet\cdots\cdots\bullet & & & & \\ & \bullet\cdots\cdots\bullet & & & \\ & & \ddots & & \\ & & & \bullet\cdots\cdots\bullet & \\ & & & & \bullet\cdots\cdots\bullet \end{array}$$

where the i–th row contains $|a_i|-1$ "points" for $i=1,\dots,l$, and the first point of each row is vertically aligned with the last point of the previous row. The point rule says that there are m columns, and the j–th column contains $|b_j|-1$ points for every $j=1,\dots,m$. For example if $-7/5=[-2,-2,-3]$ and $-7/2=[-4,-2]$ the corresponding diagram is given by

$$\begin{array}{cc} \bullet & \\ \bullet & \\ \bullet & \bullet \end{array}$$

Let Γ be either a star–shaped or a linear weighted graph. If $(\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$ admits an embedding into a standard diagonal lattice $(\mathbb{Z}^k, -\mathrm{Id})$ with basis $e_1,...,e_k$, then we will write, for every subset S of the set of vertices of Γ,

$$U_S := \{e_i \mid e_i\cdot v \neq 0 \text{ for some } v\in S\}.$$

We can now start to work towards Lemma 3.3 and Theorem 1.4.

Lemma 3.1 *Suppose* $-\frac{1}{r}=[a_1,...,a_n]$ *and* $-\frac{1}{s}=[b_1,...,b_m]$ *where* $r+s=1$. *Consider a weighted linear graph* Ψ *having two connected components,* Ψ_1 *and* Ψ_2, *where* Ψ_1 *consists of* n *vertices* $v_1,...,v_n$ *with weights* $a_1,...,a_n$ *and* Ψ_2 *of* m *vertices* $w_1,...,w_m$ *with weights* $b_1,...,b_m$. *Moreover, suppose that there is an embedding of the lattice* $(\mathbb{Z}^{n+m}, Q_\Psi)$ *into* $(\mathbb{Z}^k,-\mathrm{Id})$, *with basis* $e_1,...,e_k$, *such that* $e_1\in U_{v_1}\cap U_{w_1}$ *and* $U_\Psi=\{e_1,...,e_k\}$. *Then,*

(1) $U_{\Psi_1}=U_{\Psi_2}$, *and*

(2) $k=n+m$.

Proof We start showing that (1) implies (2). In fact, since $r+s=1$, by [12, Lemma 2.6], we have

$$\sum_{i=1}^{n}(-a_i-3)+\sum_{j=1}^{m}(-b_j-3)=-2$$

and therefore

$$|\mathrm{tr}\,(Q_\Psi)| = \sum |a_i| + \sum |b_j| = 3(n+m) - 2. \tag{3.1}$$

The matrix Q_Ψ is not singular [12, Remark 2.1] so we necessarily have $k \geq n+m$. Let us write $k = n+m+x$ for some $x \geq 0$. Since we are assuming (1), each vector of the basis $e_1, ..., e_k$ satisfies $e_i \in U_{\Psi_1} \cap U_{\Psi_2}$ and therefore $|\mathrm{tr}\,(Q_\Psi)| \geq 2k$. Moreover, since the graph Ψ has $n+m-2$ edges, it follows that $|\mathrm{tr}\,(Q_\Psi)| \geq 2k + n + m - 2 = 3(n+m) - 2 + 2x$. Hence, by (3.1), $x = 0$ and (2) holds.

We now assume that (1) does not hold and show that this assumption leads to a contradiction. Start by defining the sets $E_1 := U_{\Psi_1} \setminus U_{\Psi_2}$, $E_2 := U_{\Psi_2} \setminus U_{\Psi_1}$ and $E_{12} := U_{\Psi_1} \cap U_{\Psi_2}$. Since we are assuming that (1) does not hold, we have $E_1 \neq \emptyset$ or $E_2 \neq \emptyset$. By simmetry, we may assume that $E_1 \neq \emptyset$. It follows that there exists a smallest index $n_0 \in \{1, ..., n\}$ such that $U_{v_{n_0}} \cap E_1 \neq \emptyset$. This condition allows us to construct a new connected linear graph $\bar{\Psi}_1$ with n_0 vertices and associated string of weights $(a_1, ..., a_{n_0-1}, \bar{a}_{n_0})$, where

$$\bar{a}_{n_0} := a_{n_0} + \sum_{e_\ell \in E_1} (v_{n_0} \cdot e_\ell)^2.$$

Notice that since the intersection lattice associated with Ψ admits an embedding into a diagonal lattice, there is a naturally induced analogous embedding of the intersection lattice associated with $\bar{\Psi}_1 \cup \Psi_2$. We claim that $\bar{a}_{n_0} \leq -2$. In fact, if $n_0 = 1$ the assumption $e_1 \in U_{v_1} \cap U_{w_1}$ and the equality $v_1 \cdot w_1 = 0$ imply $|E_{12} \cap U_{v_1}| \geq 2$ and therefore in this case $\bar{a}_1 \leq -2$. On the other hand, if $n_0 > 1$ then, by definition of n_0, it holds that $U_{v_{n_0-1}} \subseteq E_{12}$. The equalities $v_{n_0-1} \cdot v_{n_0} = 1$ and $v_{n_0} \cdot w_\ell = 0$ for every $\ell \in \{1, ..., m\}$ force $|E_{12} \cap U_{v_{n_0}}| \geq 2$ and therefore $\bar{a}_{n_0} \leq -2$ as claimed.

Now, since $-2 \geq \bar{a}_{n_0} > a_{n_0}$ we have, by standard facts on continued fractions, that $-\frac{1}{\bar{r}} := [a_1, ..., a_{n_0-1}, \bar{a}_{n_0}]$ satisfies $\bar{r} + s > 1$ (since $r + s = 1$). Let $\bar{r}'$ be such that $\bar{r} + \bar{r}' = 1$. Using Riemenschneider's point diagram it is not difficult to check that $r + s = 1$ and $-2 \geq \bar{a}_{n_0} > a_{n_0}$ imply that there is some $t < m$ such that $-\frac{1}{\bar{r}'} = [b_1, ..., b_t]$. Let us call $\bar{\Psi}'_1 \subseteq \Psi_2$ the linear connected subgraph with associated string of weights $(b_1, ..., b_t)$. There are two possibilities, either $U_{\bar{\Psi}_1} = U_{\bar{\Psi}'_1}$ or $U_{\bar{\Psi}_1} \neq U_{\bar{\Psi}'_1}$.

Notice that by construction $|\bar{\Psi}_1| \leq |\Psi_1|$ and $|\bar{\Psi}'_1| < |\Psi_2|$. Moreover, if $|\bar{\Psi}_1| = 1$ [resp. $|\bar{\Psi}'_1| = 1$] then $\bar{\Psi}'_1$ [resp. $\bar{\Psi}_1$] is a (-2)–chain and it is immediate to check that in this case, since $e_1 \in U_{v_1} \cap U_{w_1}$, it holds $U_{\bar{\Psi}_1} = U_{\bar{\Psi}'_1}$. Since $\bar{r} + \bar{r}' = 1$, if $U_{\bar{\Psi}_1} \neq U_{\bar{\Psi}'_1}$ we can repeat the above construction with $\bar{\Psi}_1$ and $\bar{\Psi}'_1$ playing the

role of Ψ_1 and Ψ_2. It follows that, after a finite number of steps, we necessarily obtain from Ψ_1 and Ψ_2 two linear weighted graphs, which we still call $\bar{\Psi}_1$ and $\bar{\Psi}'_1$, such that $U_{\bar{\Psi}_1} = U_{\bar{\Psi}'_1}$ and either $\bar{\Psi}_1 \subseteq \Psi_1$ or $\bar{\Psi}'_1 \subseteq \Psi_2$. By simmetry we may assume that $\bar{\Psi}'_1 \subseteq \Psi_2$.

Since the strings of weights associated to $\bar{\Psi}_1$ and $\bar{\Psi}'_1$ are related to one another by Riemenschneider's point rule, we know, by the first part of this proof and using the same notation, that $|U_{\bar{\Psi}_1} \cup U_{\bar{\Psi}'_1}| = n_0 + t$. Consider the vector w_{t+1}. Since $U_{\bar{\Psi}_1} = U_{\bar{\Psi}'_1}$, $w_t \cdot w_{t+1} = 1$ and $w_{t+1} \cdot v_\ell = 0$ for every $\ell \in \{1, ..., n_0\}$, the vector

$$\bar{w}_{t+1} := w_{t+1} + \sum_{e_i \notin U_{\bar{\Psi}_1}} (e_i \cdot w_{t+1})\, e_i$$

satisfies $\bar{w}_{t+1} \cdot \bar{w}_{t+1} \leq -2$. It follows that the disconnected linear graph $\bar{\Psi}_1 \cup \bar{\Psi}'_1 \cup \{w_{t+1}\}$, which has $n_0 + t + 1$ vertices admits an embedding into a diagonal lattice of rank $|U_{\Psi_1}| = n_0 + t$ which contradicts [12, Remark 2.1]. $\square$

Lemma 3.2 *Let $-\frac{1}{r} = [a_1, ..., a_n]$ and $-\frac{1}{s} = [b_1, ..., b_m]$ be such that $r + s > 1$. Then there exists $n_0 \leq n$ and $m_0 \leq m$ such that $-\frac{1}{r_0} = [a_1, ..., a_{n_0}]$ and $-\frac{1}{s_0} = [b_1, ..., b_{m_0}]$ satisfy $r_0 + s_0 = 1$.*

Proof Let r' be such that $r + r' = 1$ and suppose $-1/r' = [a'_1, ..., a'_{n'}]$. Since $s > r'$, by standard facts on continued fractions there are two possibilities: either $b_i = a'_i$ for all $i \in \{1, ..., n'\}$ and $m > n'$ or there is a smallest index k such that $b_k > a'_k$. In the first case we set $n_0 = n$ and $m_0 = n'$. In the second case let us consider the first k columns of dots in the Riemenschneider's point diagram obtained from $(a_1, ..., a_n)$. Then, n_0 equals the number of rows in this diagram minus $b_k - a'_k$ and $m_0 = k$. Note that in this way $[a_1, ..., a_{n_0}]$ and $[b_1, ..., b_{m_0}]$ are related to one another by Riemenschneider's point rule and therefore $r_0 + s_0 = 1$. $\square$

Lemma 3.3 *Suppose $k \geq 3$ and $1 > r_1 \geq \cdots \geq r_k > 0$ and $r_{k-1} + r_k > 1$. Then, the intersection lattice of the plumbing associated to $Y := Y(-k + 1; r_1, ..., r_k)$ cannot be embedded into a negative diagonal standard lattice.*

Proof Let Γ be the plumbing graph of Figure 6 associated to Y, and suppose by contradiction that there exists an embedding of $(\mathbb{Z}^{|\Gamma|}, Q_\Gamma)$ into $(\mathbb{Z}^d, -\mathrm{Id})$ with basis $e_1, ..., e_d$ for some $d \geq |\Gamma|$. We will use the following notations for the vertices of Γ: v_0 for the central vertex and v^i_j for the vertices in the legs, where i indicates the leg to which v^i_j belongs and j the position in the leg, with $j = 1$ being the index of the vertex connected to the central vertex.

Since Γ has k legs connected to the central vertex which has weight $-k+1$, there must exist some basis vector, say e_1, and two legs, say L_i and L_j, such that the products $v_0 \cdot e_1$, $v_1^i \cdot e_1$ and $v_1^j \cdot e_1$ are not 0.

Let $-\frac{1}{r_i} = [a_1^i, ..., a_p^i]$ and $-\frac{1}{r_j} = [a_1^j, ..., a_q^j]$. Since $r_i + r_j > 1$, by Lemma 3.2 there exist $p_0 \leq p$ and $q_0 \leq q$ such that the strings $(a_1^i, ..., a_{p_0}^i)$ and $(a_1^j, ..., a_{q_0}^j)$ are related to one another by Riemenschneider's point rule. Moreover, since $e_1 \in U_{v_1^i} \cap U_{v_1^j}$ Lemma 3.1 applies and therefore the disconnected subgraph $\Psi \subseteq \Gamma$ consisting of the vertices $v_1^i, ..., v_{p_0}^i, v_1^j, ..., v_{q_0}^j$ satisfies $|U_\Psi| = p_0 + q_0$. Furthermore, writing $\Psi = \Psi_1 \cup \Psi_2$ where Ψ_1 [resp. Ψ_2] consists of the vertices $v_1^i, ..., v_{p_0}^i$ [resp. $v_1^j, ..., v_{q_0}^j$] we have, by Lemma 3.1 (1), $U_{\Psi_1} = U_{\Psi_2}$.

Now, since $r_i + r_j > 1$ we have $(p_0, q_0) \neq (p, q)$ so we can assume without loss of generality $p > p_0$. The vector $v_{p_0+1}^i$ satisfies $v_{p_0+1}^i \cdot v_{p_0}^i = 1$ and the equality $U_{\Psi_1} = U_{\Psi_2}$ implies that $|U_\Psi \cap U_{v_{p_0+1}}| \geq 2$. Consider the vector

$$\bar{v}_{p_0+1} := v_{p_0+1} + \sum_{e_i \notin U_\Psi} (e_i \cdot v_{p_0+1})\, e_i,$$

which by construction satisfies $U_{\bar{v}_{p_0+1}} \subseteq U_\Psi$ and $v_{p_0+1} \cdot v_{p_0+1} \leq -2$. It follows that the linear graph $\Psi \cup \{v_{p_0+1}\}$, which has $p_0 + q_0 + 1$ vertices admits an embedding into a diagonal lattice of rank $|U_\Psi| = p_0 + q_0$ which contradicts [12, Remark 2.1]. $\square$

We are now ready to prove Theorem 1.4. We restate the result for the reader's convenience:

Theorem 1.4 A closed, oriented, Seifert fibered 3–manifold of special type admits no symplectic fillings.

Proof Suppose that the oriented 3–manifold Y is orientation preserving diffeomorphic to $Y(-1; r_1, \ldots, r_k)$, where $k \geq 3$, $1 > r_1 \geq r_2 \geq \cdots \geq r_k > 0$ and $(r_1, \ldots, r_k)$ satisfies Conditions (1) and (2) of Definition 1.2. The fact that $(r_1, \ldots, r_k)$ is not realizable implies that Y is an L–space [18]. Therefore, if Y admits a symplectic filling W then $b_2^+(W) = 0$ [22, Theorem 1.4]. Consider the space $-Y = Y(1-k; \overline{r_1}, ..., \overline{r_k})$ where $\overline{r_i} := 1 - r_{k-i+1}$ for $i \in \{1, ..., k\}$. Since

$$e(-Y) := 1 - k + \sum_i \overline{r}_i = 1 - \sum_i r_i < 0,$$

by [21, Theorem 5.2] there is a negative definite plumbing graph Γ such that $-Y = \partial P_\Gamma$. Consider the 4–manifold X obtained gluing together P_Γ and W

along their common boundary. By construction X is smooth, closed and negative, therefore by Donaldson's celebrated theorem [2] its associated intersection form must be diagonalizable. It follows that if Y admitted a symplectic filling then the intersection lattice of P_Γ would admit an embedding into a diagonal, negative standard lattice. The assumption $1 > r_1 + r_2$ for Y reads $\overline{r_{k-1}} + \overline{r_k} > 1$ for $-Y$, which by Lemma 3.3 implies that the intersection lattice of P_Γ does not admit an embedding into a diagonal lattice. Therefore we conclude that Y admits no symplectic fillings. $\square$

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